ALTERNATING-CURRENT PLANEVECTOR POTENTIOMETER MEASUREMENTS AT TELEPHONIC FREQUENCIES.

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History of the Apparatus.

The measurements presented in this paper have been made in the electrical engineering laboratories of the Massachusetts Institute of Technology, with a new form of a.–c. planevector potentiometer, giving its potential readings in rectangular coördinates. This intrument lends itself readily to the measurement of vector potential differences up to frequencies of at least 2,000~. The principle of the potentiometer is the same as that described by Professor Larsen[1] in 1910; but the present form of the instrument has been worked out in the M. I. T. laboratories, mainly through the thesis studies of Mr. A. E. Hanson.[2]

The essential connections of the instrument and its method of application are outlined in Fig. 1. The anti-inductive resistance *ab* is connected in series with the primary winding *bc,* of a toroidal nonferric induction coil included in the main potentiometer circuit *P.* The secondary winding *bd,* of the induction coil, is connected at *b* with the junction of the primary coil and the resistance. The secondary winding of the induction coil, and the resistance *ab,* are both provided with suitable taps, which are brought out to dials, for adjustment of the potential connections at *k* and *l.* When a sinusoidal alternating current I_p r.m.s. amperes ∠, from the oscillator source *O,* at a suitable impressed frequency, passes through the potentiometer circuit, the alternating p.d. between the taps *k* and *l* will be

$$E_p = I_p(R + jX), \qquad \text{r.m.s. volts } \angle \quad (1)$$

where R is the resistance included between k and b in ohms, while

[1] Bibliography 9.

[2] Bibliography 13.

$jX = jM\omega$ is the mutual reactance included between l and b. M denotes the mutual inductance between the primary winding and the part bl of the secondary, and $\omega = 2\pi f$, is the angular velocity (in

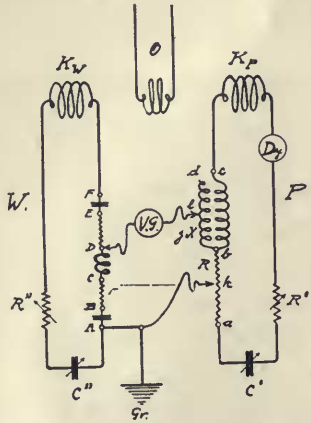

FIG. 1. Simplified diagram of connections, showing the rectangular potentiometer P arranged for exploration of the potential distribution over the working circuit W.

radians per second) of the impressed frequency f, in cycles per second.

The oscillator also supplies current to an associated working circuit W, containing the apparatus in which the distribution of planevector[3] a–c. potential is to be explored. A tuned vibration galvanometer, VG, enables a potential balance to be obtained between a selected pair of terminals on the working circuit, and the adjustable taps on the potentiometer circuit, so as to secure the relation

$$U_{AB} = I_P(R + jX), \qquad \text{r.m.s. volts } \angle \qquad (2)$$

whereby the p.d. at the terminals, say A and B, can be evaluated in terms of the calibrated constants R and X of the instrument, and the measured potentiometer current I_P, taken at standard phase. If, however, another p.d., say U_{BC}, for instance, across a standard re-

[3] A planevector may be defined as a geometrically directed complex quantity in a plane of reference, and subject to the laws of complex arithmetic, as distinguished from a vector which is subject to the laws of vector arithmetic. In this paper the term "vector" is used as an abbreviation of "planevector."

sistance r ohms, as indicated at BC, in Fig. 1, is measured immediately after U_{AB}, we have

$$U_{BC} = I_p(R_0 + jX_0), \qquad \text{r.m.s. volts } \angle \quad (3)$$

Then

$$\frac{U_{AB}}{U_{BC}} = \frac{R + jX}{R_0 + jX_0}. \qquad \text{numeric } \angle \quad (4)$$

In this way, the relative numerical values of the p.d.'s on different parts of the working circuit can be evaluated, preferably in terms of the p.d. on a standard resistance, without requiring a measurement of the potentiometer current I_p.

As thus far described, the instrument can only measure vector p.d.'s confined to a single quadrant. By means of reversing switches, however, on X and on the entire potentiometer, all four quadrants in the voltage complex plane can be explored.

A more detailed description of the instrument, and of its mode of operation, technique and limitations, appears in a paper by the writers published[4] elsewhere; so that only a brief outline of this part of the subject will here be necessary, in order to present more fully in this paper some of the results which have been secured with the instrument, at telephonic frequencies, in the laboratory.

Simple Series Circuit of Resistance, Inductance and Capacitance, at Constant Frequency and Varied Capacitance.

In Fig. 2 the connections are shown of a simple working circuit containing an inductance L of 0.8106 henry, a total resistance R of 3183 ohms, and an adjustable condenser C. Such a circuit may briefly be described as a RLC circuit. The connections pp' to the potentiometer are also indicated. The inductance L was very loosely coupled with the Vreeland oscillator *Osc*, and the oscillator frequency was maintained at 500~.

With the condenser C shorted; *i.e.*, made infinite, the potentiometer reading on 100 ohms in r was the vector OA, as obtained from rectangular coördinates in R and $-jX$, viz. $-22.5 - j176$ millivolts. This is indicated as a vector of 179.4 millivolts, at a slope

[4] Bibliography 14.

of —97°.2 with respect to the reference axis *OR*, which coincides with the phase of the current I_P in the potentiometer circuit.

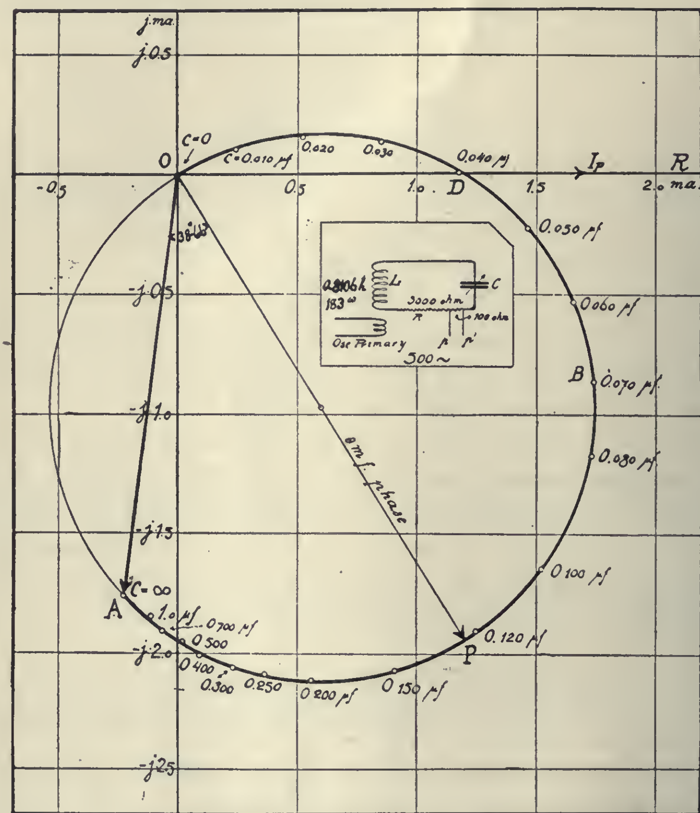

FIG. 2. Current locus, under constant impressed e.m.f. of 500~, of a simple circuit containing inductance, resistance, and capacitance, the latter being varied from 0.03 μf. to ∞.

As the capacitance in the condenser *C* was reduced, the vector voltages at the terminals *pp′* advanced along the locus *APBDO*, which is seen to be a circle passing through the origin of coördinates *R* and *jX*.

It is easy to demonstrate that a circular locus must apply to the current and therefore to the potential difference at *pp′*. The impedance in the working circuit of Fig. 2 is

$$Z = R + j\left(L\omega - \frac{1}{C\omega}\right) \qquad \text{ohms } \angle \qquad (5)$$

If we keep r, L and ω constant, but vary C, the impedance will advance vectorially along a straight line parallel to the j axis, or reactance axis. This is shown in Fig. 3, where distances along the horizontal real axis represent values of resistance R; while distances along the j axis represent values of reactance. As the value of C

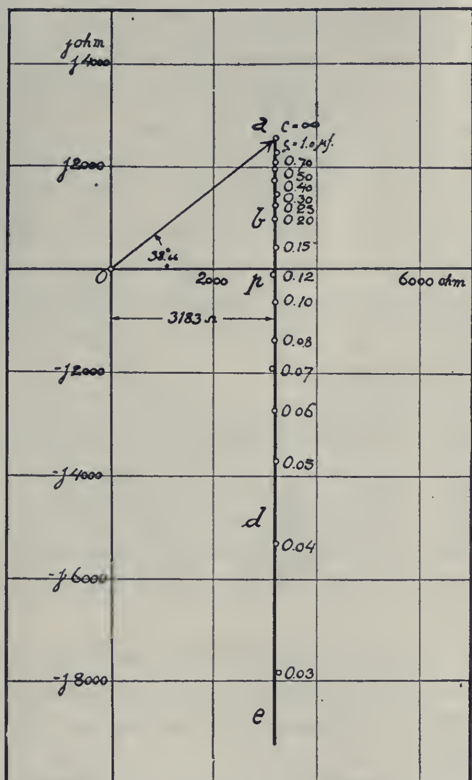

FIG. 3. Planevector impedance locus of a simple RLC circuit in which the capacitance is varied.

was changed from infinity to 0.03×10^{-6} farad, the vector value of impedance moves along the straight line $a\ b\ d\ e$. The impedance Z

therefore pursues a straight-line locus. The current in the working circuit will be

$$I_W = \frac{|E_W| \measuredangle \beta^0}{Z} \qquad \text{amperes} \angle \quad (6)$$

where $|E_W|$ is the constant size of the induced e.m.f. in that circuit, and $\beta°$ is the angle of lag of that e.m.f. behind the current in the potentiometer circuit, considered as of reference phase. This circuit happened, in this case, to be strongly condensive, owing to the adjustment of the condenser C' in Fig. 1. The angle of lag β is the angle DOP in Fig. 2. In equation (6) the vector impedance Z appears as a reciprocal, or in the denominator. It is well known that the reciprocal of any vector straight-line locus is a circular locus passing through the origin. The diameter of the circle through the origin is also the reciprocal of the perpendicular distance from the impedance origin to the impedance locus. Consequently, in Fig. 2, with $|E_W| = 7.273$ volts, the diameter OP, expressed in amperes, will be $7.273 \times 1/3181 = 0.002285$ ampere, or 2.285 milliamperes, lagging $\beta°$ behind the phase of the potentiometer current OR. The measured p.d. across 100 ohms in the resistance R will be 0.2285 volt, also lagging $\beta°$.

In reducing the capacitance of C from infinity to zero, the circular locus of potential across pp' has nearly covered three quarters of its entire circle. It is evident from a consideration of Fig. 3, that if, with the condenser shorted, additional inductance could have been inserted in the circuit, while retaining all other conditions constant, the remainder AO of the circular locus might also have been traversed.

The use of the new potentiometer thus enables this circular potential and current locus, at telephonic frequency, to be demonstrated observationally, instead of remaining on a purely abstract mathematical basis.

Simple *RLC* Circuit of Sharper Resonance.

Another example of varying the series capacitance in a simple a.-c. circuit, but at a frequency of 2010∼, is given in Fig. 4. Here almost the entire circular locus of potential and current is covered by the observations, which, shown as round dots, lie close to the circle *OABD,* except for an accidental wide unobserved gap between 0.019 and 0.022 microfarad. In this case, the resistance of the circuit was 250 ohms, and this relatively small resistance involves a relatively large change in the vector impedance near resonance, com-

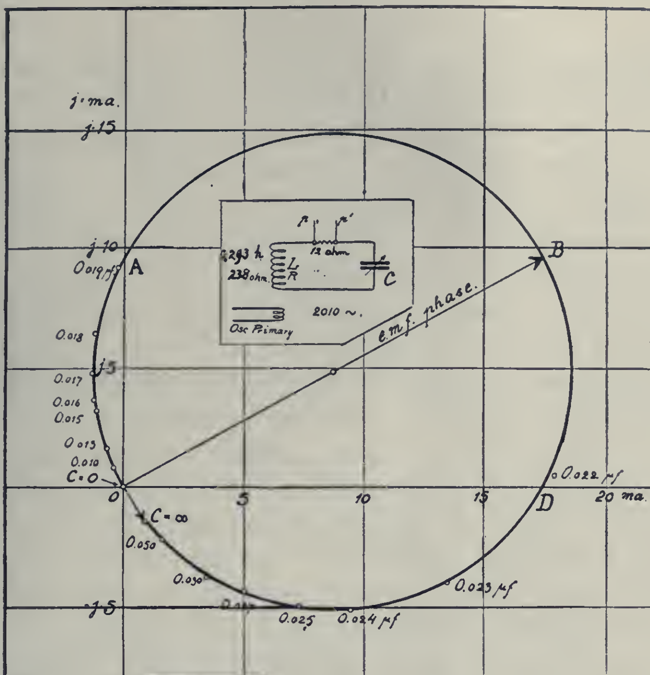

FIG. 4. Current-locus for conditions similar to those of Fig. 2, but applying to a circuit of smaller resistance, and to an impressed frequency of 2010∼.

puted at 0.0214 microfarad, with a small change of capacitance. It may be noted that in this case a change of 250 ohms in the condenser reactance changes the vector current, from resonance, through 45° difference of phase, or over a quarter of the circular locus. As the resistance R in the circuit is reduced, a correspondingly smaller change in reactance, from the resonant point, will produce this 45° change in the vector current. An a.-c. potentiometer can thus serve to measure a small capacitance with precision, by observing the vector change of current thereby produced in a sharply tuned resonant circuit of low resistance, when the condenser is inserted or removed. Other applications of the same principle will suggest themselves.

Simple *RLC* Circuit with Variations Made First in *R* Alone, and then in *C* Alone.

An example of a simple *RLC* circuit, with successive variations in resistance and capacitance, is presented in Fig. 5. Here the total resistance in the circuit could be given different assigned values between 52 and 802 ohms. The capacitance could also be varied between infinity and 0.20 microfarad. The p.d.'s were measured across taps pp', Figs. 2 or 4, and the current strengths determined therefrom. The constant inductance in the circuit was 0.1 henry. The frequency was 1006$\sim$ throughout. This produces resonance in the circuit at $C=0.25$ microfarad.

If we plot the impedance of a *RLC* circuit, like that of Fig. 4, at any constant frequency, we obtain an impedance diagram of the type represented in Fig. 6. If we maintain constant resistance in the circuit, such as $r_4=402$ ohms, and vary only the capacitance, the impedance locus will lie along the straight line AB. Such a variation of impedance, substituted in equation (6), will, as we have already seen, give rise to a circular current locus, the diameter of the circle coinciding with the vector E of impressed e.m.f. If, however, we keep the reactance constant, at some value such as $x_4=j316.2$ ohms, Fig. 6, and vary the resistance, the vector impedance will move over the straight-line locus DE. This vector impedance, inserted in (6), will also give rise to a circular locus; but the diameter of this circle will be in quadrature with the vector of impressed e.m.f. In the particular case of resonance, where the reactance is kept at zero, and the resistance is varied, the current locus will be a circle of infinite radius; *i.e.*, a straight line, coinciding with the vector of impressed e.m.f. These results are brought out experimentally in Fig. 5. It will be observed that with a capacitance of 0.25 microfarad, the locus coincides with the straight line OR. The vector impressed e.m.f. was so adjusted, by preliminary tuning of the potentiometer circuit, as to coincide with this line.

All of the circles of constant resistance have their centers on the line OR, or axis of reals.

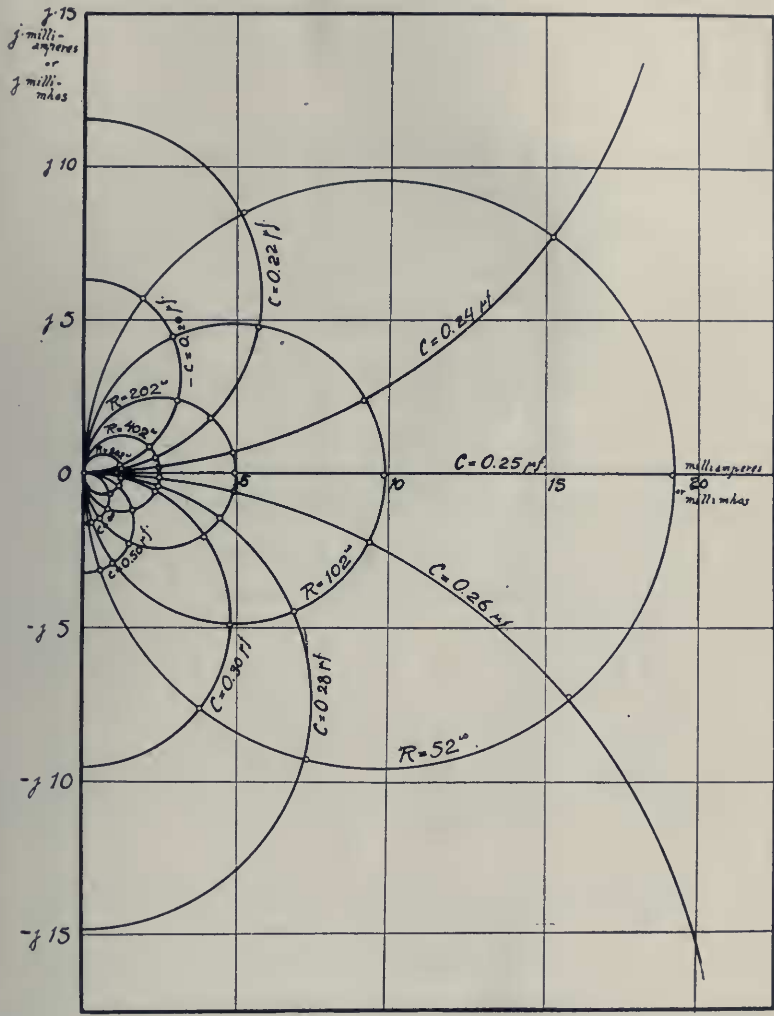

FIG. 5. Graphs of vector current or vector admittance for a RLC circuit, in which either R or C is varied alone.

In Fig. 5, the full lines are the theoretical loci and the small circles the actual potentiometer readings. Fig. 6 is obtained by inversion from Fig. 5, and shows the vector loci of impedance in the circuit, the full lines being computed. The small circles indicate the experimentally deduced results.

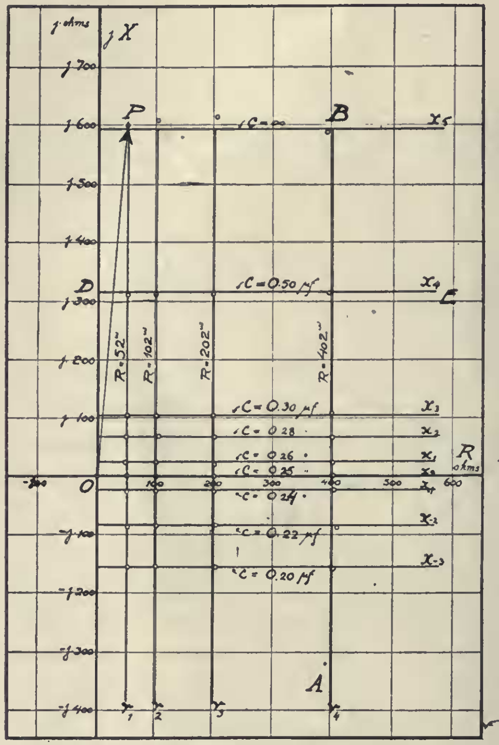

FIG. 6. Graphs of vector impedance for a *RLC* circuit, in which either *R* or *C* is varied alone.

DIVIDED CIRCUIT WITH INDUCTANCE AND CAPACITANCE IN PARALLEL.

If we take a simple *RLC* circuit, such as that shown in Figs. 2, 4 and 5, and shunt some part of the fixed inductance with an adjustable condenser, we obtain the connection diagram of Fig. 7*a*, where the inductance coil *l*, separate from the oscillator secondary, is shunted by the variable condenser *c*. Commencing with $c=0$, or the condenser removed, the vector current in the circuit, as deduced from the p.d. across the 10 ohm taps pp', was the vector $Ol=3.6$

$+j13.6 = 14.07 \angle 75°.2$ milliamperes, at 525~, Fig. 7*b*. As the capacitance in the condenser *c*, Fig. 7*a*, is increased from 0 to infinity, the current locus goes over the circular path *m l p k*, Fig. 7*b*.

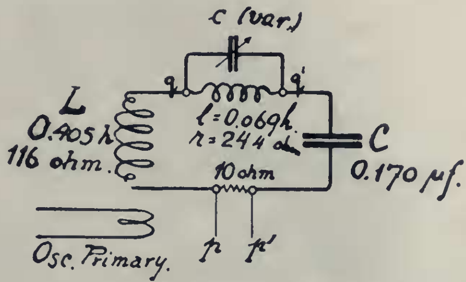

FIG. 7*a*. Circuit of inductance, resistance, and capacitance, where one part of the inductance is bridged by a variable condenser *c*.

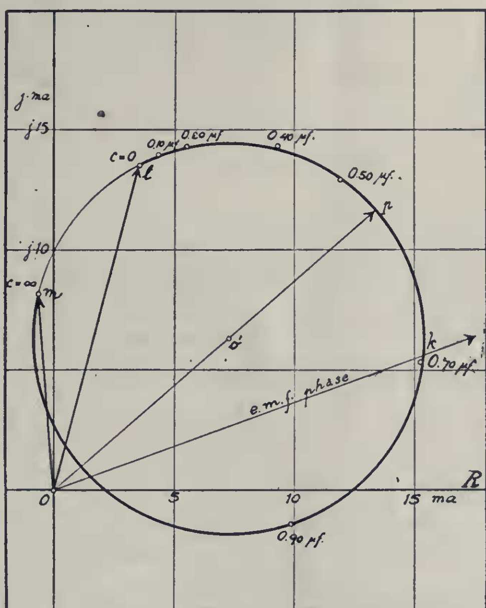

FIG. 7*b*. Locus for the main current in the circuit of Fig. 7*a*, the condenser *c* being varied from 0 to ∞, and the impressed e.m.f. being maintained constant, at 525~.

The phase of the e.m.f. induced in the coil *L*, Fig. 7*a*, is represented by the vector *Ok*, Fig. 7*b*.

The circular graph of Fig. 7*b* is a current locus to *OR* as standard current phase. The same graph may, however, be regarded as

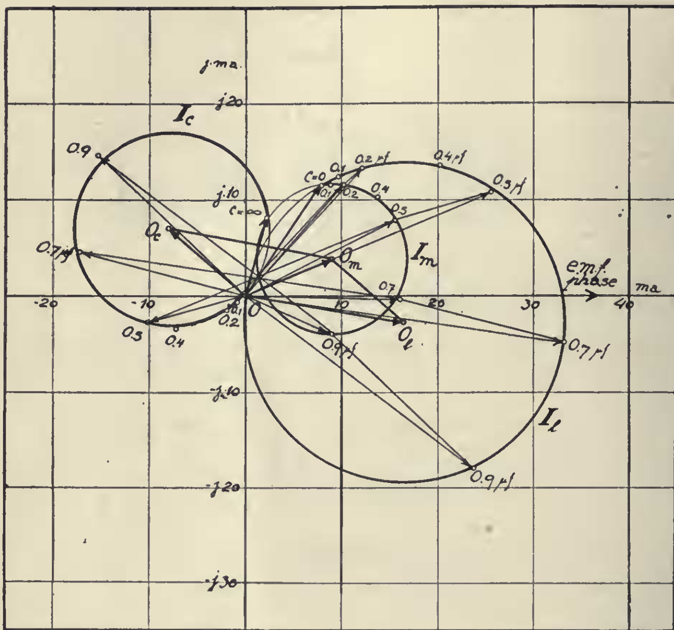

Fig. 7*c*. Vector circular loci of main and branch currents in the circuit of Fig. 7*a*, as the condenser *c* is varied.

an admittance locus to *Ok* as standard admittance phase, if we change the scale of coördinates accordingly, since

$$Y = \frac{I}{E} \qquad \text{ohms} \angle \qquad (7)$$

and E is the e.m.f. in the circuit, 3. 744 ∠ 19°.4 volts, which was kept constant both in size and in slope during the observations. If, regarding Fig. 7*b* as an admittance graph to the proper scale, we take the reciprocals of successive vector values, we obtain therefrom an impedance graph which must also be circular, according to the theory of geometrical inversion. Fig. 8 shows the vector impedance graph so obtained. It represents the vector impedance in the main circuit of Fig. 7*a* when the condenser c is varied. At $c = \infty$, the vector impedance is $OM = 464.7 \,\overline{\vee}\, 74^\circ.25$. At $c = 0$, the vector impedance is $OL = 266.2 \,\overline{\vee}\, 55^\circ.6$ ohms. The angle *mol* in Fig. 7*b* is the same (19°) as the angle *MOL* in Fig. 8. As *c* is increased from $c = 0$ to $c = \infty$, the vector impedance pursues the wide circle *LPGM*.

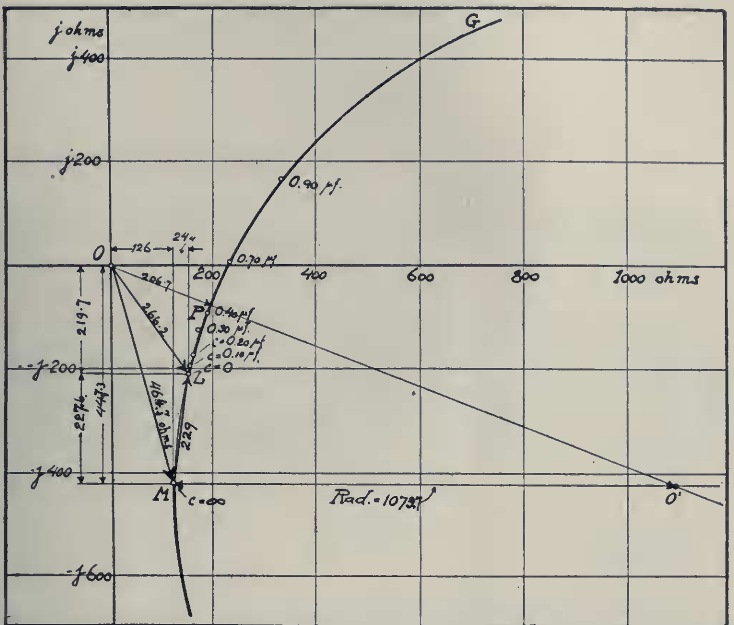

FIG. 8. Locus diagram obtained by inversion from Fig. 7*b*, and showing the variation of total main-circuit impedance of the main-circuit of Fig. 7*a*.

The explanations for these interesting circular graphs of Figs. 7*b* and 8, appear step by step in Fig. 9. Fig. 9*a* shows the vector admittance of the *l c* combination in Fig. 7*a*, as taken between the points *qq′* in the main circuit. *Ol*, Fig. 9*a*, is the fixed admittance of the coil *l*, Fig. 7*a*, at the constant impressed frequency of 525∼; namely $0.4656 - j4.343 = 4.369 \,\overline{\setminus}\, 83^\circ.53'$ millimho. The admittance of the condenser *c* will be $jc\omega$. The vector sum of *Ol* and $jc\omega$ will evidently lie on the straight line *lm*, as *c* varies from 0 to ∞. The corresponding impedance of the *lc* combination, between points *qq′* in the main circuit, Fig. 7*a*, will be the vector reciprocal of the straight line locus *lm*, Fig. 9*a*, and must therefore be a circle, as has been already pointed out. The impedance of the combination will vary over the circle *LUSM*, Fig. 9*b*, from *OL* with $c=0$ to $OM=0$ with $c=\infty$. The diametral maximum impedance *OR* or 2147

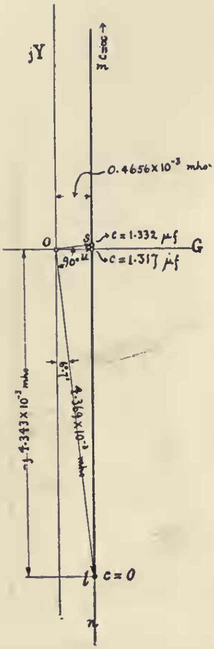

FIG. 9*a*. Vector admittance of *l* and *c* in parallel as *c* is varied from zero to infinity.

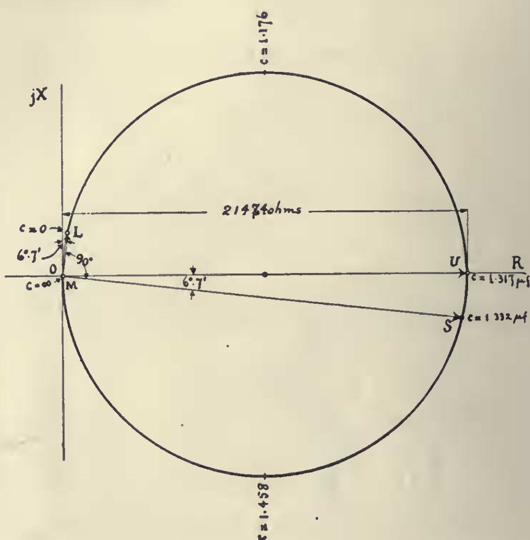

FIG. 9*b*. Vector impedance of *l* and *c* in parallel as *c* is varied from zero to infinity. Vector inversion of Fig. 9*a*.

ohms, occurs at the value of *c* (1.317 μf) at which *Ou*, Fig. 9*a*, is horizontal, and at which value the vector admittance in 9*a* has minimum size. At this value of *c*, the admittance of the *lc* combination is a minimum, its impedance a maximum, and the p.d. at terminals *qq'* in phase with the main current. This condition occurs when the susceptance of the condenser $jc\omega$ is equal and opposite to the susceptance of the coil. Thus

$$y=\frac{1}{r+jl\omega}=\frac{r-jl\omega}{r^2+l^2\omega^2}=\frac{r}{r^2+l^2\omega^2}-j\frac{l\omega}{r^2+l^2\omega^2}=g-jb \qquad \text{mhos} \angle \qquad (8)$$

so that

$$b = \frac{x_l}{|z^2|} = \frac{l\omega}{r^2 + l^2\omega^2} = \frac{1}{l\omega + \dfrac{r^2}{l\omega}} \qquad \text{mhos.} \quad (9)$$

Consequently, at minimum admittance and maximum impedance,

$$c\omega = \frac{1}{l\omega + \dfrac{r^2}{l\omega}} \qquad \text{mhos} \quad (10)$$

or

$$c = \frac{1}{l\omega^2 + \dfrac{r^2}{l}} \qquad \text{farads.} \quad (11)$$

It may be noted that the condition of maximum-impedance in the lc combination, and which has been defined as "parallel resonance," corresponding to zero total susceptance, differs from the condition of simple series resonance within the lc circuit, which occurs when the total reactance is zero, or when

$$c\omega = \frac{1}{l\omega} \qquad \text{mhos.} \quad (12)$$

This condition is found in the diagram, Fig. 9*a*, at the total vector admittance *os*, when the capacitance in c is 1.332 μf. This occurs when the angle *sol*, Fig. 9, is 90°, or when the angle *uso* is equal to the angle *uol* of the coil's admittance. The angle *sou* is 6°.7′, the complement of *uol*.

In Fig. 9*b*, the impedance *OS* of the lc combination is that which is presented at series resonance. It has the value $2135 \nwarrow 6°.7'$ ohms. The angle *SOU* is the same as the angle *sou*. At the capacitance $c = 1.317$ μf., the p.d. at qq', on the lc combination, will be in phase with the main current.

If we increase the impedances of Fig. 9*b* by the fixed impedance in the remainder of the circuit, Fig. 7*a*, we obtain the total circuit impedance as shown in Fig. 9*c*, where OM' is the vector impedance $126 - j447.3 = 464.7 \nwarrow 74°.16'$. If we add to this fixed impedance, OM', the vector circle *LUS* of Fig. 9b, we obtain the resultant vector locus $M'K'H'U'Q'$ of Fig. 9*c*. As the capacitance in c varies from 0 to ∞, the impedance in the main circuit varies from OL' to OM', around the circular arc $H'Q'$.

The vector $OP' = 206.7 \,\overline{\backslash}\, 20^\circ.26'.49''$ ohms, is the point of apparent resonance or minimum impedance in the main circuit; *i.e.*, maximum main current, as the result of adjusting condenser c. It is not a true resonant point, since there is neither cophase, zero resistance, zero susceptance nor equality of undamped frequencies.

The vectors OK' and OH' are two cophase vectors, at each of which the current in the main circuit is in phase with the main impressed e.m.f. In general, there will be either two such cophase points, or none. A single cophase point may, however, present itself as an intermediate case. Such cophase points are, in general, not to be classed as resonance points.

The vector points, U' and S', corresponding to U and S in Fig. 9*b*, and also to u and s in Fig. 9*a*, are only of secondary signification in relation to the main-circuit impedance. That is to say, although they correspond to local-circuit resonances in lc, they do not, in general, represent main circuit resonances.

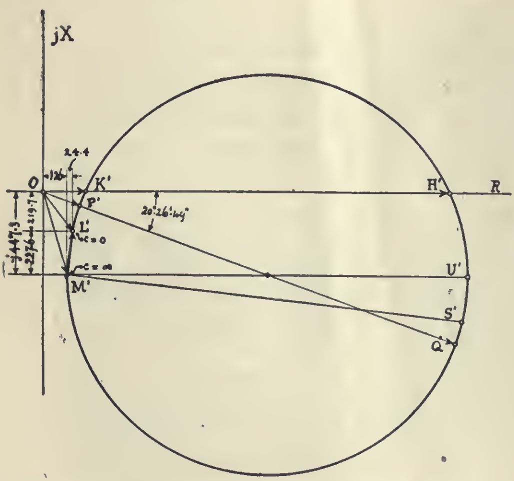

FIG. 9*c*. Total impedance of circuit including l and c in parallel, as obtained from the vector addition of OM to the locus of Fig. 9*b*.

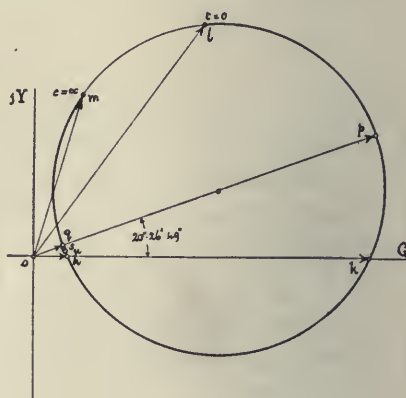

FIG. 9*d*. Total admittance of circuit as obtained by inversion from Fig. 9*c*.

Finally, the vector OQ' represents the maximum impedance of $2357 \nwarrow 20°.26'.49''$ ohms. It is in cophase with the vector OP of apparent resonance.

If we invert the impedance graph $L'P'H'Q'$ with respect to the origin O, by plotting reciprocals of the corresponding vectors, we obtain the total admittance graph of Fig. 9*d*. This pursues the circular locus *lpkhm,* as c varies from 0 to ∞. This may also be regarded as a main current graph to a suitable scale of amperes, with the voltage at reference phase OG. The vector admittance and current will have a maximum at Op, the point of apparent resonance, cophase points at k and h, minimum at q and local internal resonances of the lc loop at u and s.

Fig. 7*b* shows the circular locus observations in the main current corresponding to Fig. 9*d*; while Fig. 8, deduced from 7*b*, corresponds to Fig. 9*c*. It is therefore brought out experimentally, with the new potentiometer, what has perhaps hitherto been known only in abstract theory, that the variation of pure reactance in a branch circuit gives rise to circular-locus current variations in the main circuit.

The experimental case, presented in Fig. 7, corresponds to that of a radio receiving system, in which L and C correspond to the antenna path to ground. The loop circuit lc then corresponds to an oscillation circuit conductively connected with the antenna, c being tuned to bring about maximum antenna current.

Composition and Resolution of Vector Circular Loci.

The vector loci of current in the fixed inductance l and the condenser c, as the latter is adjusted from zero to infinity, are presented in Fig. 7*c*. These take the form of circles marked respectively I_l and I_c. The vector sum, I_m, or main current in the circuit LC of Fig. 7*a*, is reproduced from Fig. 7*b* to e.m.f. standard phase. It may be noted that in these three circles the maximum or diametral values of current strength occur at different condenser settings. I_m has its maximum near $c = 0.5\,\mu$f., I_l near $0.7\,\mu$f., and I_c near $0.9\,\mu$f.

It has already been shown that the circular variation of impedance in the condenser, by reason of its adjustment, gives rise to a current of circular locus in the main circuit. The p.d. at the terminals qq' must also have a circular locus, as it consists of the vector difference between the constant impressed e.m.f. E, and a vector drop of circular locus due to a circular-locus current in the constant impedance of L and C. Consequently, the fixed branch l must receive a current of circular locus. The current in the variable condenser branch must also have a circular locus, as will be seen from the following relations. The main current I is

$$I = \frac{E}{Z + \dfrac{1}{g + jc\omega}} \qquad \text{amperes } \angle \qquad (13)$$

$$= \frac{E(g + jc\omega)}{(1 + Zg) + jZc\omega} \qquad \text{amperes } \angle \qquad (14)$$

where g is the fixed admittance of the coil l or $1/(r + jl\omega)$ in mhos $\angle$, and Z is the fixed impedance, in ohms $\angle$, of the main LC circuit, outside of the lc combination.

This main current I has already been shown to have a circular locus, in reference to Fig. 9, and the conclusion is confirmed algebraically from (14), by Möbius' theorem, as will be further discussed in the appendix.

The branch current in the condenser I_c is

$$I_c = I\frac{jc\omega}{g + jc\omega} = \frac{Ejc\omega}{(1 + Zg) + jZc\omega} = \frac{E}{Z - j\dfrac{1 + Zg}{c\omega}} \qquad \text{amperes } \angle \qquad (15)$$

In its last form, the expression for I_c indicates the inversion of a straight line, which shows that I_c has a circular locus, passing through the origin. The branch current in the fixed inductance l is

$$I_l = I\frac{g}{g + jc\omega} = \frac{Eg}{(1 + Zg) + jZc\omega} \qquad \text{amperes } \angle \qquad (16)$$

also evidently a circular locus when c varies.

Consequently, by analysis based on purely electrical relations, the vector circular locus I_m is resolved into a pair of component vector circular loci I_l and I_c, such that at any assigned value of the capacitance c, the vector sum of I_l and I_c, corresponding thereto, is equal to the value of I_m then existing. Moreover, the vector sum of OO_c and OO_l, which are respectively the vectors from the origin to the centers of the branch current circles, is equal to OO_m, the vector to the center of the main current circle.

This process of splitting a main current vector circular locus into branch current circular loci can be indefinitely extended, by subdividing the admittance of any one branch into any desired number of equivalent branch admittances. Thus, by following the known electrical rules of splitting admittances, a vector circular locus, followed by the main current, can be resolved into any desired number of component vector circular loci.

Conversely, when a number of branch circuits, forming divisions of a main circuit containing a steady e.m.f., connect a pair of terminals, and one of them has its impedance varied circularly, the currents in all will undergo circular variation in locus. The vector sum of all the currents in the unvaried branches can be obtained by replacing their several fixed admittances by a single joint admittance, and determining, from electrical considerations, the vector circular locus established in this joint admittance by the circular variation of impedance in the outstanding branch.

Voltage Distribution Along a Series of High-Resistance Coils at 2010~.

A series of high-resistance coils, such as a megohm box, divided into a number of sections, develops an interesting electrical condition, when used with alternating currents, which does not obtrude itself upon the attention of the observer when such resistances are used in continuous-current tests. This condition is a superposed alternating-current distribution in the coils, due to their distributed capacitance and the relatively high a.-c. potentials which are impressed upon their terminals. In the continuous-current case, the capacitances and applied potentials are present, but the charges are fixed, and these do not interfere with the testing current. The a.-c. phenomenon has been noticed and studied, but does not appear to have been presented in the light of potentiometer measurements.

The connections for the test are indicated in Fig. 11. The high-resistance box *AB* of 100,000 ohms, is divided into 10 coils of 10,000 ohms each. The end *B* is grounded, and also connected to the resistance tap of the potentiometer. The exploring tap, passing

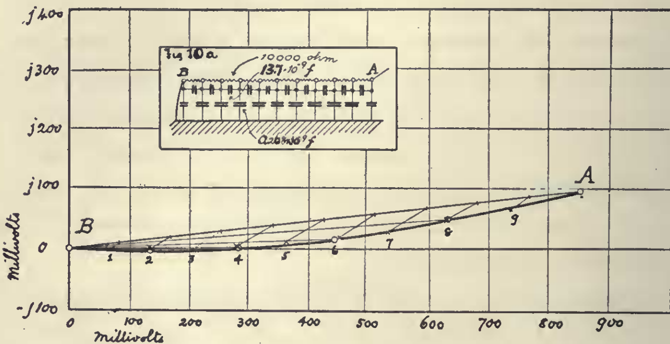

FIG. 10. Vector voltage drop observed along a sectioned high resistance box of 100,000 ohms in 10 coils of 10,000 ohms each, at the frequency of 2,010 cycles per second. Crosses indicate observed, and circles computed values.

through the vibration galvanometer, makes contact in succession with the junctions 1, 2, 3 . . . 10 of the resistance coils, Fig. 11.

The results of an exploration, at 2010~, are presented in Fig. 10. With the potential of the grounded point *B* at the origin, the potential at *A* was $855 + j98$ millivolts. If the fall of potential along the coils were regular, the intermediate potentials should have fallen on the straight line *AB,* whereas they actually fell upon the curve *B,* 1, 2, 3 . . . *A.* The vector deviations are indicated at each junction, and they affect both size and slope. The reason for this behavior evidently is that, as indicated in Fig. 10*a,* the resistance box resembles an artificial line, in which the line sections are resistances shunted by condensers; while each section has a condenser leak to ground. The observations indicate that each 10,000-ohm section subtends, at this frequency, an average hyperbolic angle of $0.13 \angle 15°$, which would be accounted for by a shunting condenser of 13.7 millimicrofarads per section, and a leak condenser of 0.268 millimicrofarad. The results indicate that the drop of potential in the section nearest to the ground point is very distinctly less than one tenth of the total drop *BA,* and that in the sections near *A* very distinctly greater. Consequently, when using this subdivided resistance as a volt box, the *B* coils give too small, and the *A* coils too large a result.

It may be observed that it is impossible completely to avoid the introduction of these virtual shunting and leak condensers, in the construction of high-resistance boxes for a.-c. tests. If these vir-

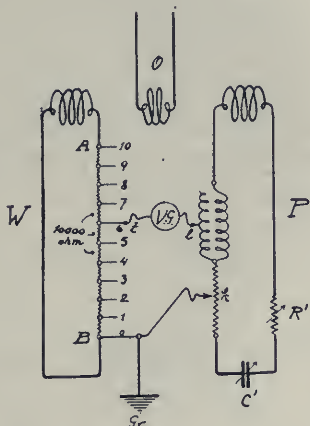

FIG. 11. Exploration, by means of the potentiometer *P*, of the voltage drop along a high-resistance box *AB*, of 100,000 ohms.

tual condensers are dissymmetrical, and vary from coil to coil, it becomes very difficult to make a proper correction for the error due to charging currents. If, however, care is taken to preserve symmetry of structure throughout the resistance box, so that the distributions of virtual capacitance remain uniform from coil to coil, the correction may be readily computed, at any impressed frequency, by hyperbolic functions, according to the well-known theory of artificial lines, from a single set of observations for determining the line constants. A knowledge of the error will thus virtually eliminate the error. Suitably supported and connected metallic ground shields, surrounding each resistance coil, would serve to distribute the leak capacitance uniformly. High-resistance section boxes should, therefore, be designed so as to subtend small hyperbolic angles per section; but whatever the section angle is, it should be regular throughout.

Receiving-End Impedance of a Cable with a Variable-Inductance Load.

The a.-c. potentiometer lends itself advantageously to the experimental study of a.-c. artificial lines in the laboratory. An example of such an application appears in Fig. 12. The plan of connections shows an artificial telephone cable ten miles (16.1 km.) in length, in *I*-sections of two miles each, or less. The lumpiness correction for this cable, at 1200∼, was found to be negligible. The constants of the cable are $r''=88$ ohms per loop mile (54.7 ohms per loop km.) and $c''=0.06$ microfarad per loop mile (0.0373 μf. per loop km.). The total conductor resistance was thus 880 ohms, and the total distributed loop capacitance 0.6 microfarad.

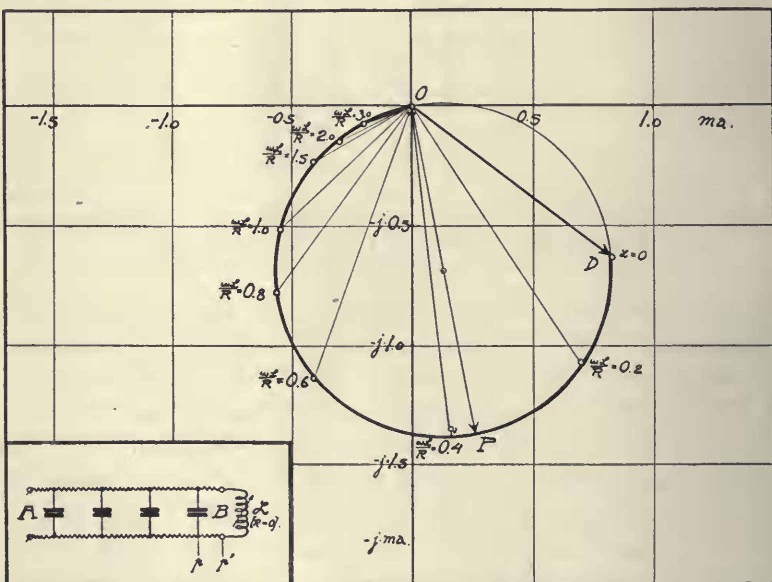

Fig. 12. Vector current circular locus at receving end of an artificial telephone cable with variable inductance load, under constant impressed e.m.f. at sending end and at a steady frequency of 1,208 cycles per second.

At the sending end *A* of the artificial cable, a pliotron oscillator impressed a steady e.m.f., reduced to 1.0 volt at reference phase and of frequency 1208∼. At the receiving end *B'*, an adjustable inductance L, of negligible resistance, was used as a receiving-end load. The received current was measured by observing the p.d. across the last resistance section pp' of the artificial line.

With the inductance L reduced to zero, the receiving-end current was observed to be the vector $OD = 1.03 \times 10^{-3} \overline{\diagdown}\, 37^\circ$ milliampere. The formula for I_B is

$$I_B = \frac{1.0 \angle 0^\circ}{z_0 \sinh \theta + jL\omega \cosh \theta} \qquad \text{amperes} \angle \qquad (17)$$

where z_0 is the surge impedance of $439 \overline{\diagdown}\, 43^\circ.6$ ohms,

θ is the angle subtended by the line $2.0 \angle 44^\circ.3$ hyps $\angle$,

ω is the impressed angular velocity $= 2\pi \times 1208 = 7590$ radians/sec.

With $L = 0$, I_B becomes $1.035 \times 10^{-3} \overline{\diagdown}\, 37^\circ.4$ ampere.

By giving to $jL\omega$ successively increasing values up to $j2640$ ohms, the observed received current strength was found to pursue the circular locus DPO. This circle has its diameter at OP, when the value of the reactance is 352 ohms. At this condition, the received current strength was a maximum at $1.4 \times 10^{-3} \overline{\diagdown}\, 80^\circ$ ampere.

The sector of the circle in Fig. 12 between O and D might have

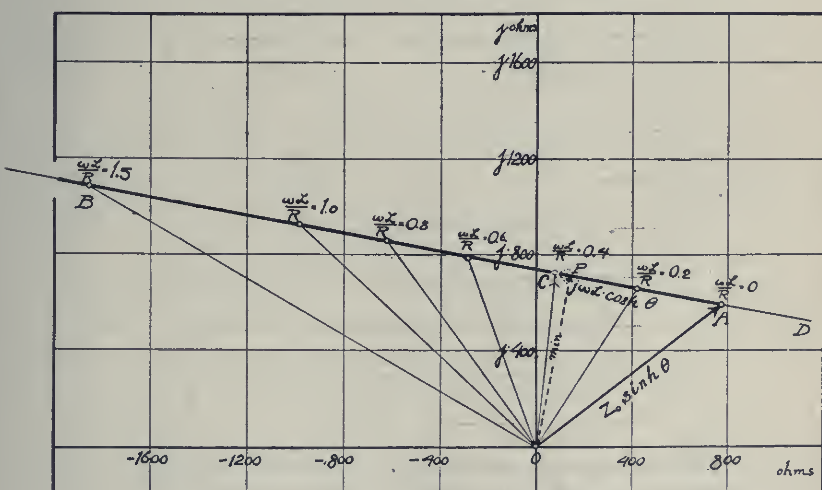

FIG. 13. Rectilinear graph of receiving-end impedance for case presented in Fig. 12, as the load of receiving-end inductance is varied, without changing the resistance from zero value.

been covered, if a condenser had been substituted for the inductance load.

Fig 13 shows why a circular current graph might be expected in this case at the receiving end. The denominator in formula (17) contains the vector sum of a fixed impedence $z_0 \sinh\theta$ and the variable impedance $jL\omega\cosh\theta$. In this term ω and $\cosh\theta$ are constant, so that L is the only variable. The vector OA, Fig. 13, represents $z_0 \sinh\theta$. AC is the vector $jL\omega\cosh\theta$, for L corresponding to $L\omega/R = 0.4$. The vector sum $z_0\sinh\theta + jL\omega\cosh\theta$ is then OC. To successive values of L, from $L = 0$ to $L = 0.175$ henry, correspond successive distances from A, along the straight line AB. The vector locus of the sum appearing in the denominator of (17), is therefore a straight line. The reciprocal of this vector sum must consequently follow a circular locus, and corresponds to the circular locus of Fig. 12. OP is the perpendicular from the origin to the line AB. The point P thus marks a minimum receiving-end impedance, and a maximum received current. This corresponds to a certain type of resonance.

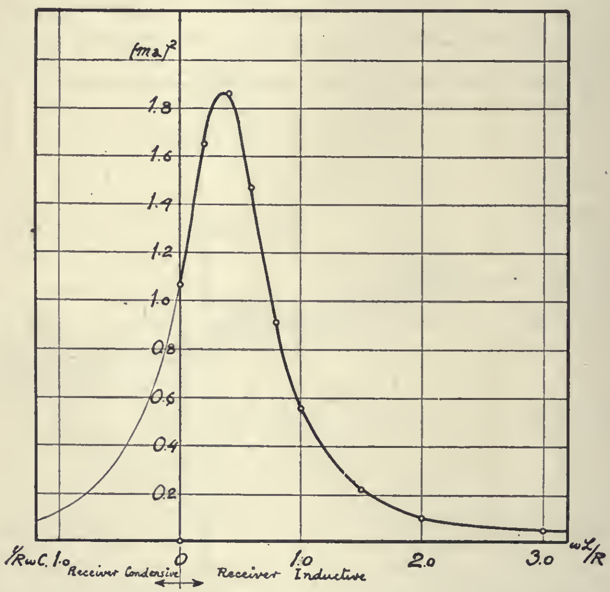

FIG. 14. Resonance curve of received current strength squared against the ratio of receiving end reactance to total resistance.

If we plot the square of the received current strength of Fig. 12, as scalar ordinates, to $L\omega/R$ as abscissas, we obtain the resonance curve of Fig. 14. It follows from the linear relations of $L\omega/R$ along the straight line *AB*, in Fig. 13, that the ordinates of the resonance curve in Fig. 14 will lie symmetrically at equally spaced abscissas on each side of the maximum. If a condenser had been used as a receiving-end load beyond the point $L = 0$, the part of the resonance curve, Fig. 14, on the left-hand of the origin, might have been covered.

Prevalence of Circular Vector Loci.

It may be noted that in the cases of Figs. 2, 4, 5, 7, 9 and 12, circular graphs of vector potential loci have presented themselves, and many measurements in the laboratory, at constant frequency, lead to such circular graphs.

If we consider any fixed network of conductors, we may assume that all joints are electrically good, all leaks are steady, and all iron cores removed from the inductances; or, in other words, assume that each and all of the elements of the network obey Ohm's law.

Let us select arbitrarily any pair of terminals *A B* in the network, at which a constant simple alternating e.m.f. is applied, in the steady state, at a fixed frequency, and also select any other pair of terminals *C D* in the network, at which the effects of the applied e.m.f. are to be looked for. It will be shown, in the Appendix, that if we introduce, between *C* and *D*, an impedance which varies along a circular locus in the impedance plane, a straight line variation being included as a particular case, then both the current and voltage will be caused to vary along a circular locus at *C D*. Moreover, the current and sending-end impedance at the terminals *A B* will be caused to vary along a circular locus. Again, if a circularly varied impedance is inserted between the impressed e.m.f. and the *A B* terminals, the voltage and current at those terminals will be caused to vary circularly, and also the voltage and current at *C D*. A further development of this remarkable phenomenon will be found in the Appendix. It means that a conducting network has a wonderful propensity for reproducing impressed circular variations. A network of *n* elements, in one of which a circular variation of impedance or admittance is impressed, will give rise to n-1 new circular variations of current; *i.e.*, one in each of the other elements, assuming all impressed e.m.f.'s simultaneously acting on the system remain constant. The circular variations in terminal *p.d.'s* throughout the system will be of the order n^2, neglecting intermediate potentials between terminals.

APPENDIX.

Circular Variation in Networks of Conductors.

It is proposed to establish the following main proposition:

> In any network of conductors, all the elements of which obey generalized Ohm's law, subjected in the steady state to any frequency of constant e.m.f. (including zero frequency as the limiting continuous-current case), circular variation of the impedance of any element will produce a corresponding circular variation in the current in every element of the network. Moreover, the p.d. between any two points of the network will be caused to vary circularly. The sending-end impedance or admittance between any two points on the network will be likewise caused to vary circularly. By "circular variation" is meant variation over a planevector circular locus, including the straight-line locus as a limiting case.

The prevalence of circular loci in relation to alternating-current circuits has been recognized in the literature of the subject,[5] which contains various scattered theorems bearing upon special cases. A few of these theorems may advantageously be collated here in presenting the demonstration of the main proposition.

It was shown by Clerk Maxwell[6] that if, in any continuous-current network, two terminals, say A and B, are selected as sending-end terminals, and two other terminals, say C and D, are selected as receiving-end terminals, a current I applied to the network at AB would produce the same p.d., at CD, as would be produced at AB, if the same current I were applied at CD. This theorem may be

[5] Bibliography, 5, 6, 7, 7*a*, 10.

[6] Bibliography, 3; also 4, 8.

given an interesting interpretation in terms of the equivalent T of the network, with regard to terminals $A\,B$, $C\,D$. By generalization from several well-known examples of equivalent T-s, LaCour[7] demonstrated that any a.-c. or d.-c. network can be replaced by a certain equivalent T, in general dissymmetrical, so that this T, insofar as regards conditions at A, B, C, D will be the exact equivalent of the actual network.

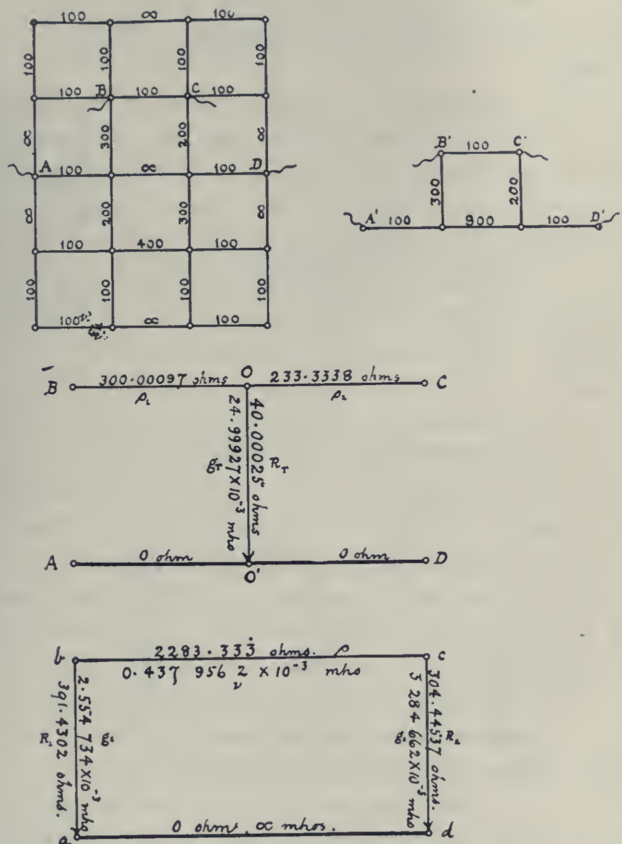

FIG. 15. Continuous-current network with its equivalent T and Π.

As a simple example, we may consider the continuous-current plane network of Fig. 15. The resistances of the various elements in ohms are marked thereon. Some of them are made infinite, in

[7] Bibliography, 7 and $7a$.

order to simplify the computation. With respect to the pairs of terminals *A B* and *C D*, this network reduces to the *A′ B′ C′ D′* connection of six elements. By any of the known processes, this may be reduced to the equivalent *T* of Fig. 15*b*, in which the staff resistance is 40.00025 ohms. It is evident that 1 ampere applied to this T at the *A B* terminals, will produce 40.00025 volts at *C D*, and that reciprocally 1 ampere applied to this *T* at the *C D* terminals will also produce 40.00025 volts at *B A*. The "mutual impedance"[8] of the network between the pairs of terminals *B A* and *C D* is thus 40.00025 ohms.

Maxwell also showed[9] that if continuous e.m.f. were applied at *A B*, and a current thereby produced through the terminals *C D* when short-circuited, the same e.m.f. applied at *C D* would produce the same current through the terminals *A B* short-circuited. A similar theorem had been enunciated already by Kirchhoff.[10]

This proposition may, in a similar way, be given an interpretation in terms of the equivalent Π of the network. In Fig. 15*c*, *b c a d* is the equivalent Π with respect to the terminals *A B*, *C D*. It is evident that an e.m.f. of 1 volt applied at *b a* would produce a current of 0.4379562 milliampere at *c d*, and that reciprocally, 1 volt at *c d* would produce the same current at *a b* shorted. In other words, the mutual admittance of the network with respect to the two selected pairs of terminals is the architrave admittance of the equivalent Π. Moreover, the equivalent Π of mutual admittance is the conjugate system of the equivalent *T* of "mutual impedance," and may be directly computed therefrom. Consequently, if the "mutual impedance" *T* of a network, with respect to two pairs of terminals, has been ascertained, the mutual admittance Π of the same network and terminals can be deduced by Π–*T* substitution and without further experimental investigation.[11]

In Fig. 16, a relatively simple alternating-current plane network example is worked out. The impedance of each element of the nine-element network is marked thereon. The selected terminals are *A B* and *C D*. The equivalent *T* is shown in Fig. 16*b*, and the

[8] Bibliography 12.
[9] Bibliography 3, 4, 8.
[10] Bibliography 1, p. 32.
[11] Bibliography 6*a*.

equivalent Π in Fig. 16*c*. The "mutual impedance" of the network for these pairs of terminals is $10.48605 \angle 95^\circ.00'.47''$ ohms, and the mutual admittance $0.936265 \times 10^{-3} \angle 34^\circ.59'.31''$ mho.

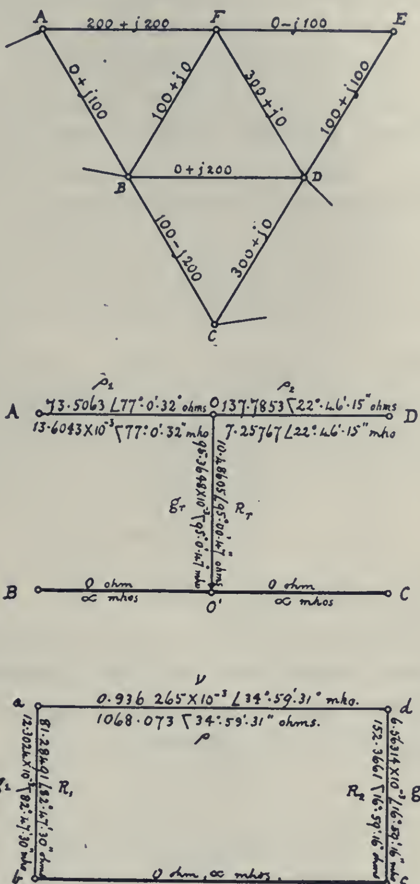

FIG. 16. Alternating-current network with its equivalent T and Π.

In any actual network, it is theoretically easy to determine the elements of the equivalent T experimentally. It suffices to measure E_{CD} and I_{AB} in order to determine R_T. A measurement of the sending-end impedances at each pair of terminals in turn, will then serve to evaluate r_1 and r_2. Similar treatment applies to the experimental determination of the Π, if this is preferred.

In the equivalent T's of Figs. 15 and 16, all of the series impedance appears in the upper arms, and none in the lower arms. All of the impedance might, however, be transferred to the lower arms, without altering the equivalence of the circuit. Moreover, any desired share of impedance might be transferred from the upper to the lower arms, according to the rules of T and I equivalent circuits.[12] Similarly, the architrave impedance of the Π's of Figs. 15 and 16 may be either wholly or partly transferred to the lower line.

In any dissymmetrical T and its conjugate Π, we have the relation

$$\frac{\rho_1}{\rho_2} = \frac{R_1}{R_2} = \frac{g_2}{g_1} \qquad \text{numeric } \angle \qquad (18)$$

In the case of Fig. 15, this ratio is 1.2857. In the case of Fig. 16, it is $0.53348 \angle 99°.46'.47''$. This also means that in any a.-c. case, the difference in the slopes of ρ_1 and ρ_2 will be the same as the difference in the slopes of R_1 and R_2, or

$$\bar{\underline{\rho_1}} - \bar{\underline{\rho_2}} = \bar{\underline{R_1}} - \bar{\underline{R_2}} = \bar{\underline{g_2}} - \bar{\underline{g_1}} \qquad \text{degrees} \qquad (19)$$

this difference being $99°.46'.47''$.

Again in any dissymmetrical T and its conjugate or equivalent Π, the geometrical mean of the two T-arm impedances, times the architrave admittance equals the geometric mean of the two Π-leak admittance times the T-staff impedance, or

$$\nu\sqrt{\rho_1\rho_2} = R_\tau\sqrt{g_1 \cdot g_2} \qquad \text{numeric } \angle \qquad (20)$$

In the case of Fig. 15, this product is 0.11587. In that of Fig. 16, it is $0.094224 \angle 62°.6'.40''$. From this relation it also follows that

$$\bar{\underline{R_\tau}} + \bar{\underline{g_1}} = \bar{\underline{\nu}} + \bar{\underline{\rho_2}} \qquad \text{degrees} \qquad (21)$$

In the case of Fig. 16, this quantity is $12°.13'.16''$.

Returning now to the main proposition, if an alternating-current network is connected, at the receiving-end terminals $D\,C$, to a circularly varied impedance load, the impedance of the circuit $ODCO'$,

[12] Bibliography 11 and 11*a*.

Fig. 16*b* will also have circular variation, and the admittance of this branch, including this load, must also vary circularly by the geometry of inversion.[13] If, then, we add the constant admittance of the staff leak g_T, Fig. 16*b*, it follows that the total admittance on the right-hand side of OO' will have circular variation, under circular variation of the load at BC. Taking the reciprocal of this circular admittance, the impedance of the system on the right-hand side of OO' will also have circular variation. Adding to this the constant impedance AO, the total sending-end impedance at terminals AB must have circular variation, as likewise the sending-end admittance.

Consequently, a circular variation of load at DC must produce a circular variation of current at AB under constant impressed e.m.f., or a circular variation of voltage at AB under constant impressed current.

Again, referring to Fig. 16*c*, if the constant e.m.f. impressed at *ab* is E, and an admittance load y, which varies circularly, is applied at terminals *dc*, then the admittance at *ab*, excluding the constant leak *ab* is the circularly varying admittance:

$$Y = \cfrac{1}{\rho + \cfrac{1}{g_2 + y}} \qquad \text{mhos } \angle \qquad (22)$$

Hence the entering current in the architrave *ad* will be

$$I = EY = \frac{E(g_2 + y)}{1 + \rho(g_2 + y)} \qquad \text{amperes } \angle \qquad (23)$$

Of this current, the fraction $y/(g_2 + y)$ will pass through the load and the current i in this load will therefore be

$$i = \frac{E \cdot y}{1 + \rho(g_2 + y)} \qquad \text{amperes } \angle \qquad (24)$$

The ratio E/i is the receiving-end impedance of the loaded system and is

$$Z = \rho + \frac{1 + \rho g_2}{y} \qquad \text{ohms } \angle \qquad (25)$$

[13] Bibliography 2.

In this expression, if y varies circularly, so does its reciprocal, and therefore so does Z. Consequently, if the sending-end impressed voltage is held constant, and y is varied circularly, the current delivered to the load at the receiving-end will undergo a circular variation. From similar considerations, it may be seen that if the sending-end impressed current is held constant, the voltage at receiving terminals will also undergo circular variation, when y is varied circularly.

We have hitherto considered the effect of a circularly varied load at the receiving terminals CD, in producing circular variations of impedance, voltage, and current, both at those terminals and at the sending terminals AB. We may now consider, in like manner, the effect of circularly varied impedance, voltage and current at the sending end.

Referring to Fig. 16b, let the receiving terminals DC be connected through a fixed impedance load σ ohms $\angle$, such that

$$\rho_2 + \sigma = z_2 = \frac{1}{y_2} \qquad \text{ohms } \angle.$$

Then the total admittance at O will be $g_r + y_2$ mhos $\angle$. The total impedance at the terminals AB is then $\rho_1 + [1/(g_T + y_2)]$ ohms $\angle$. Let an e.m.f. of fixed frequency and constant vector value E be applied at the terminals AB, through an impedance z_1 ohms $\angle$, which impedance is varied circularly. Then the total impedance Z_A of the circuit at the sending-end is

$$Z_A = z_1 + \rho_1 + \frac{1}{g_\tau + y_2} \qquad \text{ohms } \angle \quad (26)$$

Since z_1 is supposed to be the only variable, the circular variation of z_1 causes circular variation in Z_A; so that the current entering the network at AB is circularly varied under constant e.m.f. E; or, if the entering current should be maintained constant, the impressed voltage E must be varied circularly to correspond.

The entering current at AB being

$$I_A = \frac{E}{Z_A} = \frac{E(g_\tau + y_2)}{(z_1 + \rho_1)(g_\tau + y_2) + 1} \qquad \text{amperes } \angle \quad (27)$$

The current I_c passing through the constant load σ at $C\,D$ is

$$I_c = I_A \cdot \frac{y_2}{y_2 + g_\tau} = \frac{E \cdot y_2}{(z_1 + \rho_2)(g_\tau + y_2) + 1} \qquad \text{amperes} \angle \quad (28)$$

The denominator in this expression is a vector which undergoes circular variation, when z_1 is varied circularly, and since Ey_2 is constant, I_c must also vary circularly. This means that the receiving-end impedance of the network at CD, with respect to constant e.m.f. at AB, undergoes circular variation as z_1 is varied circularly.

If constant current and not constant e.m.f. is impressed at AB, through a circularly varied impedance z_1, a similar expression for receiving-end current and voltage at CD is obtained.

Finally, if with z_1 and σ constant, the impressed e.m.f. and current are varied circularly at AB, it is manifest that the p.d. and the current in each and every element of the network, including the load at CD, will be varied circularly and in the same simple proportion. In this case, the circular variations reproduced are all of the same type throughout the network, whereas in the preceding cases discussed, the circular variations produced are, in general, different in the different elements.

We have hitherto considered only the pairs of terminals $A\,B$ and $C\,D$. Since, however, these are chosen arbitrarily in the network, the above propositions must apply between any pairs of terminals; so that if the impedance load at any pair of receiving terminals is varied circularly, the impedance of the system, including the load, will be caused to vary circularly at any other pair of terminals. Moreover, if either the current or the voltage impressed at any pair of terminals is varied circularly, the potential and current at any other pair of terminals will be caused to vary circularly. The case of constancy, or zero variation, must be included as belonging to a circular locus of zero radius. Zero variations will occur if the terminals $A\,B$ are "conjugate" to the terminals $C\,D$.[14]

Finally, the current in any element of the network must vary circularly when a constant e.m.f. is impressed at any pair of terminals and any other element has its admittance, or impedance, circularly varied. This is seen from an examination of Kirchhoff's or

[14] Bibliography 1 and 3.

Maxwell's demonstration,[15] when the reasoning is extended to complex quantities. It may be seen that in the symbols here employed,

$$i_{rs} = E_{pq} \cdot \frac{A + By}{C + Dy} \qquad \text{amperes } \angle \qquad (29)$$

where i_{rs} is the current in the element rs, when the e.m.f. E_{pq} is inserted in the element pq, and y is the admittance of any third element uv, which is subject to circular variation; while A, B, C, and D are system constants, independent of y. It is well known, from the theory of functions of a complex variable, that the linear transformation represented by (29) gives rise to a circular locus, if y varies along a circle.

It may be noted that by reason of the known principle of vector superposition of currents in a network, the multiplication of circular loci extends to cases where a plurality of e.m.f.'s coexist. The proposition may also be extended to include mutual inductances. Strictly speaking, iron-cored inductances are excluded, however, owing to the imperfect application of Ohm's law to them, under varying permeability and magnetic skin effect.

If the impressed alternating e.m.f. or e.m.f.'s are impure, so that harmonics are present, any rectilinear variation in the impedance or admittance of any element in the network will also be rectilinear for any harmonic frequency, and so will give rise to circular variations in the harmonic voltages and currents throughout the network as well as in the fundamental. In other words, the multiplication of circular variations in a network, due to rectilinear variation in the impedance or admittance of one element, applies not only to the fundamental frequency, but also to any superposed frequencies.

Summary.

1. Experimental results obtained with a new type of a.-c. potentiometer are discussed.

2. The circular graphs of current and admittance are analyzed, for the case of a simple RLC circuit, at the successive constant frequencies of 500, 1,000 and 2,010 cycles per second, when either R or C is varied alone.

[15] Bibliography 1, p. 25 and 3, Vol. 1, Section 282*a*.

3. The circular graphs of current, admittance, voltage and impedance are analyzed for the case of a divided circuit, operated at constant e.m.f. and frequency. The problem of composition and resolution of component and main circular graphs is discussed in the light of the experimental results.

4. The distribution of potential over a sectional high-resistance box is explored, at an impressed frequency of 2,010 cycles per second.

5. The circular graph of receiving-end current is given for the case of an artificial telephone cable operated at constant voltage, and at the frequency of 1,208 cycles per second, when the receiver has variable purely reactive impedance.

6. Certain general propositions are offered concerning the development of vector circular loci in the voltage, current, impedance and admittance of an alternating current network in any steady state. These propositions are shown to be connected with the equivalent T or Π of the network, with respect to any two pairs of selected terminals.

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